VARIANCE COMPONENT ESTIMATION: PROOF OF EQUIVALENCE
OF ALTERNATIVE METHOD 4 TO HENDERSON'S METHOD 3

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Abstract

An alternative way of carrying out the generalized form of Henderson's Method 2 for estimating variance components is shown equivalent to Henderson's Method 3.

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Zelen, in the discussion of Searle (1968) states that alternative Method 4 given there is equivalent to Method 3. Proof of this is now given.

The model being considered is

$$y = \mu 1 + X_1\beta_1 + X_2\beta_2 + X_3\beta_3 + X_4\beta_4 + e$$

We assume that $\beta_f = \beta_1$ and $\beta_r = (\beta_2 \quad \beta_3 \quad \beta_4)$ are the fixed and random effects respectively. Now in Method 3, the sum of squares for fitting β_3 alone would be

$$R(\beta_3)_3 = y'X_3(X_3'X_3)^{-}X_3'y. \tag{1}$$

The alternative Method 4 is to apply Method 3 directly to z, where

$$z = Wy \tag{2}$$

$$W = I - X_f(X_f'X_f)^{-}X_f' \tag{3}$$

$$W = W' = W^2. \tag{4}$$

The model for z is

$$z = WX_r\beta_r + We$$

$$= WX_2\beta_2 + WX_3\beta_3 + WX_4\beta_4 + We.. \tag{5}$$

Therefore, applying (1) to this, we get, for fitting just β_3 alone (i.e. reduced model $z = WX_3\beta_3 + We$)

$$R(\beta_3)_5 = z'(WX_3)[(WX_3)'WX_3]^-(WX_3)'z$$

$$= z'WX_3(X_3'WX_3)^-X_3'Wz$$

$$= y'W^2X_3(X_3'WX_3)^-X_3'W^2y$$

$$= y'WX_3(X_3'WX_3)^-X_3'Wy.$$

To show that this is identical to applying generalized least squares to the model $z = WX_3\beta_3 + We$, we first recall generalized least squares for the general model

$$y = X\beta + e, \text{ with var } (e) = V.$$

The sum of squares is

$$R(\beta) = y'V^-X(X'V^-X)^-X'V^-y.$$

The analogy with $z = WX_3\beta_3 + We$, where $\text{var}(We) = W$ is

$$R(\beta_3)_{GLS} = z'W^-(WX_3)[(WX_3)'W^-WX_3]^-(WX_3)'W^-z$$

$$= y'WW^-WX_3(X_3'WW^-WX_3)^-X_3'WW^-Wy$$

$$= y'WX_3(X_3'WX_3)^-X_3'Wy$$

$$= R(\beta_3)_5.$$

Therefore the alternative Method 4 (Method 5) is equivalent to generalized least squares.

We now show that this Method is also equivalent to Method 3 applied to y. To do this we utilize, from (1)

$$R(\beta_f)_3 = y'X_f(X_f'X_f)^{-1}X_f'y$$

$$R(\beta_f\beta_3)_3 = y'\left[(X_f \quad X_3)\begin{pmatrix} X_f'X_f & X_f'X_3 \\ X_3'X_f & X_3'X_3 \end{pmatrix}^{-}\begin{pmatrix} X_f' \\ X_3' \end{pmatrix}\right]y$$

and so

$$R(\beta_3|\beta_f) = R(\beta_f\beta_3)_3 - R(\beta_f)_3$$

$$= y'(X_f \quad X_3)\left[\begin{pmatrix} X_f'X_f & X_f'X_3 \\ X_3'X_f & X_3'X_3 \end{pmatrix}^{-} - \begin{pmatrix} (X_f'X_f)^{-} & 0 \\ 0 & 0 \end{pmatrix}\right]\begin{pmatrix} X_f' \\ X_3' \end{pmatrix}y$$

Write

$$Q = X_3'X_3 - X_3'X_f(X_f'X_f)^{-1}X_f'X_3 = X_3'WX_3.$$

Then from the generalized inverse of a partitioned matrix

$$R(\beta_3|\beta_f) = y'(X_f \quad X_3)\left\{\begin{bmatrix} (X_f'X_f)^{-} + (X_f'X_f)^{-}X_f'X_3Q^{-}X_3'X_f(X_f'X_f)^{-} & -(X_f'X_f)^{-}X_f'X_fQ^{-} \\ -Q^{-}X_f'X_f(X_f'X_f)^{-} & Q^{-} \end{bmatrix} - \begin{bmatrix} X_f'X_f & 0 \\ 0 & 0 \end{bmatrix}\right\}\begin{pmatrix} X_f' \\ X_3' \end{pmatrix}y$$

$$= y'(X_f \quad X_3)\begin{bmatrix} (X_f'X_f)^{-}X_f'X_3 \\ -I \end{bmatrix}Q^{-}[X_3'X_f(X_f'X_f)^{-} \quad -I]\begin{pmatrix} X_f' \\ X_3' \end{pmatrix}y$$

$$= y'[X_f(X_f'X_f)^{-}X_f'X_3 - X_3]\,Q^{-}[X_3'X_f(X_f'X_f)X_f' - X_3']y$$

$$= y'(-W)X_3Q^{-}X_3'(-W)y$$

$$= y'WX_3(X_3'WX_3)^{-}X_3Wy$$

$$= R(\beta_3)_5.$$

Hence Method 3, on z, gives $R(\beta_3)_5$ which is identical to Method 3 on y giving $R(\beta_3|\beta_f)$. This seems to suggest some kind of optimality for Method 3.

Reference

Searle, S. R. (1958) Another look at Henderson's methods of estimating variance components. Biometrics 24, December.